# Complex susceptibility + propagation of light

Plan: (1) propagation of light in medium

(2) visualization of susceptibility

(3) complex structure + the Kramers-Kronig relation

Last time:

$$\chi \equiv P/\varepsilon_0 E = \frac{\rho q^2}{m\,\varepsilon_0}\left[\frac{1}{\omega_0^2 - \omega^2 + i\Gamma\omega}\right]$$

## Propagation of light in medium – use of the susceptibility

Maxwell's equations in medium:

(1) $\frac{\partial D}{\partial t} = \nabla \times H$ where $D \equiv \varepsilon_0 E + P = \varepsilon_0 (1+\chi) E$

$H = B/\mu_0 - M = B/\mu_0$

(2) $\frac{\partial B}{\partial t} = -\nabla \times E$

In our case ther is no magnetization: $M \equiv 0$

We can obtain the wave equation by taking the time derivative of (1) and using (2)

$$\frac{\partial}{\partial t}\, \varepsilon_0 (1+\chi) \frac{\partial E}{\partial t} = \nabla \times \frac{1}{\mu_0} \frac{\partial B}{\partial t}$$

$$\mu_0 \varepsilon_0 (1+\chi) \frac{\partial^2 E}{\partial t^2} = \nabla \times \left[-\nabla \times E\right] = \nabla^2 E \qquad [\text{since } \nabla \cdot E = \rho = 0]$$

$$\boxed{\frac{1+\chi}{c^2} \frac{\partial^2 E}{\partial t^2} = \nabla^2 E}$$

This is the Maxwell wave eqn.

To see the solution let us write

$$\chi = \chi' + i\chi''$$

Im part

Re part

We expect a solution of a wave-like form:

$$E(x,t) = E_0\, e^{i(kx - \omega t)}$$

Plugging this form into the wave equation we obtain:

$$(1+\chi'+i\chi'')\frac{\omega^2}{c^2}\cancel{E_0 e^{i(kx-\omega t)}} = k^2 \cancel{E_0 e^{i(kx-\omega t)}}$$

Hence we find a relation between $k$ & $\omega$:

$$k^2 = (k'+ik'')^2 = k'^2 - k''^2 + 2ik'k'' = \frac{\omega^2}{c^2}(1+\chi'+i\chi'')$$

$$k'k'' = \frac{\omega^2}{2c}\chi''$$

If $\chi$ is complex so is $k$!

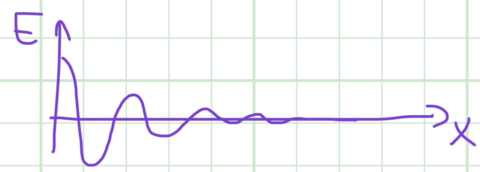
After some algebra:

$$k' = \frac{\omega}{\sqrt{2}c}s \qquad s^2 = 1+\chi'+\sqrt{(1+\chi')^2+\chi''^2}$$

$$k'' = \frac{\omega}{\sqrt{2}c}\frac{\chi''}{s}$$

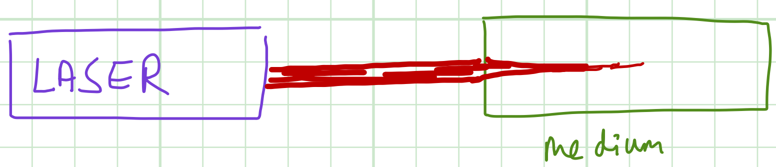
Since $k'' \sim \chi''$ it is reasonable to think that $\chi''$ has to do with absorbtion. We will see more later today.

$$E = E_0 e^{i(k_R x-\omega t)} e^{-k_I x}$$




Question: why did we choose $k$ to be complex but not $\omega$?

Answer: the underlying physics problem is that of a laser shining onto our dissipative material, hence $\omega = \omega_{laser}$ is real.




Question: what is the index of refraction?

Answer: $c_{medium} = \omega/k = c/\sqrt{1+\chi'+i\chi''}$

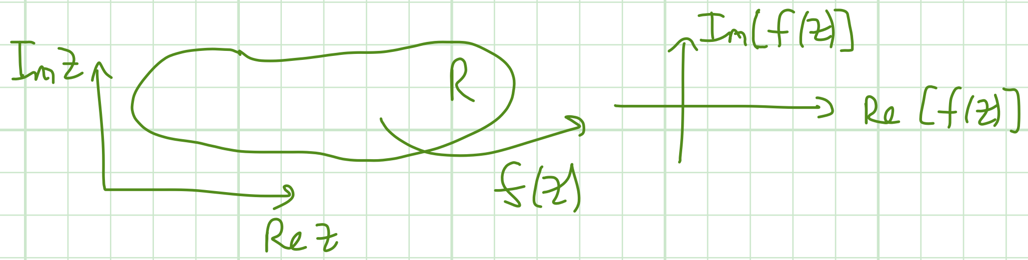
$$n = c/c_{medium} = \sqrt{1+\chi'+i\chi''}$$

# Kramers – Kronig relation

## The analytic structure of susceptibilities

We need some results from complex analysis

(1) $f(z)$ is an analytic function if it is inf. differentiable over the region $R$, i.e. it is smooth over $R$.




Examples of analytic functions (over the complex plane)

$$f(z) = z \qquad f(z) = z^2 \qquad f(z) = e^z$$

Examples of non-analytic functions

$$f(z) = 1/z \qquad f(z) = |z| \qquad f(z) = z^*$$

(2) Cauchy's integral formula

$$\frac{1}{2\pi i} \oint_{\partial R} \frac{f(z)}{z-a}\, dz = f(a) \qquad \text{if } a \in R$$

This formula relies on the Cauchy's integral theorem

$$\oint f(z)\, dz = 0$$

Sketch of proof:

$$\oint f(z)\, dz = \oint (u+iv)(dx+idy) = \oint (u\,dx - v\,dy) + i\int (v\,dx + u\,dy)$$

now apply Green's theorem to conver line integral into a surface integral $\left[ \oint A\, dl = \int \nabla \times A\, ds \right]$

$$\oint (u\,dx - v\,dy) = \iint dx\,dy \left[ -\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right] = 0 \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$i\oint (v\,dx + u\,dy) = \iint dx\,dy \left[ \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right] = 0$$

by Cauchy – Riemann

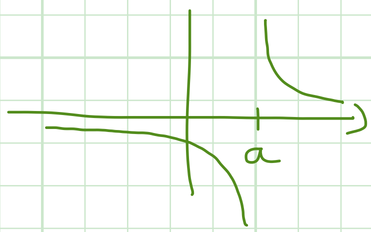
(3) Dirac formula [ should be applied to an integral ]

$$\frac{1}{x-x'-i\varepsilon} = P\frac{1}{x-x'} + i\pi\,\delta(x-x')$$

here P means the Principal value

$$P\int f(x)\,dx = \lim_{\varepsilon\to 0}\left[\int_{-\infty}^{x'-\varepsilon} + \int_{x'+\varepsilon}^{\infty}\right] f(x)\,dx$$

Example: $P\int \frac{1}{x-a} = 0$




$$P\int \frac{1}{x-a}\sin(x)\,dx = \pi\cos(x)$$

↑ use principalValue → True in Mathematica

Analytic structure of $\chi(\omega)$

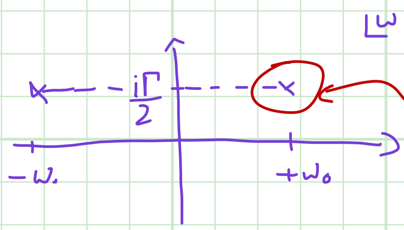
$$\chi(\omega) = \frac{C}{\omega_0^2-\omega^2 + i\Gamma\omega} = \frac{C}{(\omega_0+\omega)(\omega_0-\omega) + i\Gamma\omega}$$

poles of $\chi(\omega) \Rightarrow (\omega_0+\omega)(\omega_0-\omega) + i\Gamma\omega = 0$

$$\omega = \frac{1}{2}\left[i\Gamma \pm \sqrt{4\omega_0^2 - \Gamma^2}\right]$$

Draw on side

⇒




focus on this pole to simplify calc.

⇒ Two poles in the upper half-plane.

⇒ If $\omega_0 \gg \Gamma$ ⇒ [O.S. limit]

$$\omega = \frac{1}{2}i\Gamma \pm \omega_0 + O(\Gamma^2)$$

why are the poles of $\chi$ in the upper half-plane but not in the lower half plane?

$$P(\omega) = \chi(\omega)\,\varepsilon_0 E(\omega)$$

Fourier transforming we find :

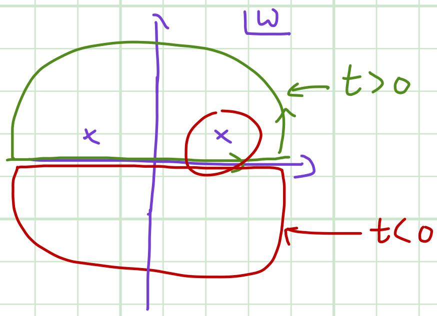
is this step clear ?

$$\int e^{i\omega t}\frac{d\omega}{2\pi} P(\omega) = P(t) = \int e^{i\omega t}\frac{d\omega}{2\pi}\chi(\omega)\,\varepsilon_0 E(\omega) = \int_{-\infty}^{t} dt'\,\chi(t-t')\,\varepsilon_0 E(t')$$

So $\chi$ relates $P(t)$ to $E(t')$ at a previous time.

where $\chi(t) \equiv \int \frac{d\omega}{2\pi}\, e^{i\omega t}\chi(\omega) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega t}}{[\omega_0 - \omega + i\Gamma/2]} \frac{C}{2\omega_0}$

in our case




$\int_{-\infty}^{\infty} d\omega = \oint_C d\omega = 2\pi i \sum \text{Residues}$

$\omega = a + ib$

$e^{i\omega t} = e^{iat} e^{-bt}$

So if $t>0$, we use the upper contour, that encircles the pole(s) in $\chi(\omega)$

for $\begin{cases} t>0 \\ b \to +\infty \end{cases}$ $e^{i\omega t} \to 0$

generic: $\sum$ Residues

our case: $\oint_C d\omega \frac{e^{i\omega t}}{\omega_0 - \omega + i\Gamma/2}\left(\frac{C}{4\pi\omega_0}\right) = 2\pi i \left(\frac{C}{4\pi\omega_0}\right) e^{i\omega_0 t - \Gamma/2\, t}$

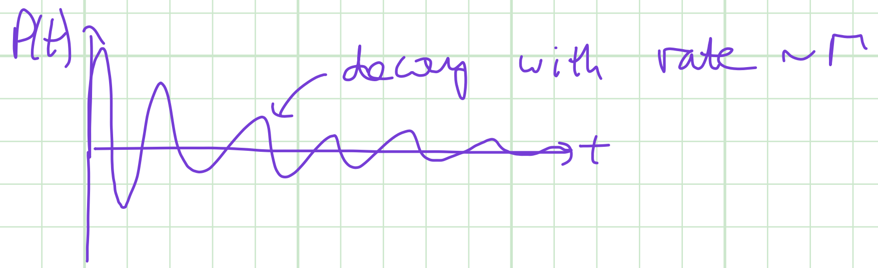
On the other hand if $t<0$, we use the lower contour

for $\begin{cases} t<0 \\ b \to -\infty \end{cases}$ $e^{i\omega t} \to 0$

generic: $\oint_C d\omega\, e^{i\omega t}\chi(\omega) = 0$ since the lower contour does not encircle any poles

Hence: $\chi(t) = i\Theta(t)\frac{C}{2\omega_0} e^{i\omega_0 t - \Gamma/2\, t}$ (the $\Theta(t)$ is generic)

So if $E(t) = E_0\,\delta(t)$ we find that $P(t)$ has a "ring-down"




(1) if $\chi$ would have poles in the lower half-plane the ringing in $P(t)$ would precede the impulse from $E(t)$

Thus violating causality [this would be ring-up that is terminated by E]

(2) if $\Gamma \to -\Gamma$ then instead of ring-down we would get ring-up

$\Rightarrow$ this is OK in driven systems, e.g. lasers.

⇒ If the medium is compose of a collection of oscillators, each with its own susceptibility $\chi_\alpha$, then the total susceptibility is

$$\chi = \sum_\alpha \chi_\alpha$$

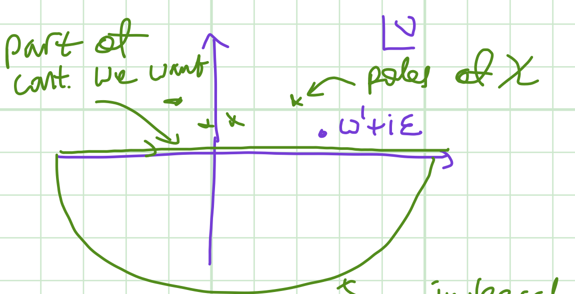
Kramers - Kronig: relation between $\chi'$ and $\chi''$

Consider the integral

$$\int_{-\infty}^{\infty} d\omega \, \frac{\chi(\omega)}{\omega' - \omega + i\varepsilon}$$

where $\varepsilon \to 0^+$ and $\omega' \in \mathbb{R}$

We can perform the integral by contour integration




integral over this contour is zero

[generically $\chi(\omega) \sim \frac{1}{\omega}$ for $|\omega| \to \infty$]

$$\int_{-\infty}^{\infty} d\omega \, \frac{\chi(\omega)}{\omega' - \omega + i\varepsilon} = \oint d\omega \, \frac{\chi(\omega)}{\omega' - \omega + i\varepsilon} = 0$$

Now let's apply the Dirac formula

$$0 = \int_{-\infty}^{\infty} \frac{d\omega \, \chi(\omega)}{\omega' - \omega + i\varepsilon} = P\int_{-\infty}^{\infty} d\omega \, \frac{\chi(\omega)}{\omega' - \omega} + i\pi \int_{-\infty}^{\infty} d\omega \, \delta(\omega - \omega')\, \chi(\omega)$$

$$= P\int_{-\infty}^{\infty} d\omega \, \frac{\chi(\omega)}{\omega' - \omega} + i\pi \chi(\omega')$$

Hence: $\chi(\omega) = \frac{1}{i\pi} P \int_{-\infty}^{\infty} d\omega' \, \frac{\chi(\omega')}{\omega' - \omega}$ Providing a relation between the real and imaginary parts of $\chi$

Explicitly:

$$\chi'(\omega) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\chi''(\tilde{\omega})}{\tilde{\omega} - \omega} \, d\tilde{\omega}$$

$$\chi''(\omega) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\chi'(\tilde{\omega})}{\tilde{\omega} - \omega} \, d\tilde{\omega}$$